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Modal Expansion of Structural Systems with Time Delays

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Conventional modal analyses are not applicable to structural systems with time delays because time delays destroy any possible orthogonality among system eigenvectors. A new series solution method is proposed for transient response analysis of a class of delayed linear structural systems. In this method, inverse Laplace transform is combined with a root locus sensitivity analysis, leading to a relation between the transfer function residues and eigensolutions of the delayed system. With this relation, closed-form time-domain response can be accurately and efficiently estimated in series of eigenvectors, without involving transfer function singularities and s-domain integration, which are usually encountered in inverse Laplace transform. The proposed method is illustrated on a gyroscopic dynamic system under delayed feedback control.

Nomenclature

$H(t)$	= Green's function
$\hat{H}(s)$	= transfer function
T_d	= time delay
$\boldsymbol{u}_k, \boldsymbol{v}_k$	= system eigenvectors
$Z(s)$	= impedance matrix
λ_k	= system eigenvalue
$\mu_j(s)$	= impedance eigenvalue
σ_k	= root locus sensitivity coefficient
$\phi_j(s), \psi_j(s)$	= impedance eigenvectors

I. Introduction

MODAL expansion or eigenfunction expansion is a widely used tool in engineering analysis. Since Rayleigh[1] pioneered it for classical conservative systems, modal analysis has been extensively studied for various dynamic systems. Caughey and O'Kelly[2] decoupled the equations of motion of proportionally damped systems using real matrices formed by the corresponding undamped eigenvectors. Foss[3] presented a complex-valued modal analysis for nonproportionally damped systems. Meirovitch[4] obtained a stationary principle for the eigenvalue problem of rotating structures. Huseyin[5] and Meirovitch[6] used complex state-space eigenvectors to give modal series solutions for general nonself-adjoint mechanical systems. Inman[7] and Inman and Olsen[8] examined similarity transformations for eigenfunction expansions of asymmetric lumped and distributed parameter systems. Yang[9] studied eigenfunction expansion of constrained/combined damped systems by using the eigensolutions of the original equations of motion. More recently, Yang and Xu[10] found modal series solutions of one-dimensional non-self-adjoint distributed dynamic systems with eigenvalue-dependent boundary conditions.

This work is concerned with modal series solutions for structural systems with time delays. Time delays may be encountered in machines with hydraulic and pneumatic components or with transportation lags,[11] systems under digital controls where the execution of numerical operations takes time,[12] and active vibration controllers with time delay parameters purposely introduced as part of control mechanisms.[13,14] The time delays in those systems destroy any possible orthogonality relations among system eigenvectors. Hence, existing modal analysis techniques are invalid for closed-form estimation of the response of delayed structural systems.

In this paper, a modal expansion for a class of delayed linear structural systems is proposed. The method, called the Z-prime method (ZPM), is developed through combination of inverse Laplace transform with a root locus sensitivity analysis. The major thrust of the method is to establish a relation between the transfer function residues and eigensolutions of the delayed system. With this relation, closed-form transient response can be practically and accurately estimated in series of eigenvectors, without involving in transfer function singularities and s-domain integration, which are usually encountered in inverse Laplace transform. Unlike existing modal analysis techniques, the ZPM does not depend on any orthogonality relations.

II. Statement of Problem

The structural system in consideration is an n-degree-of-freedom (n-DOF) dynamic system governed by the delay differential equation

$$\left(A_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A_1\frac{\mathrm{d}}{\mathrm{d}t} + A_0\right)\boldsymbol{y}(t) + \left(E_1\frac{\mathrm{d}}{\mathrm{d}t} + E_0\right)\boldsymbol{y}(t - T_d) = \boldsymbol{f}(t) \quad (1)$$

with the initial conditions

$$\boldsymbol{y}(0) = \boldsymbol{a}_0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y}(0) = \boldsymbol{b}_0 \quad (2)$$

where $\boldsymbol{y}(t)$ and $\boldsymbol{f}(t)$ ($\in R^n$) are the vectors of generalized coordinates and external disturbances, respectively, and $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ are given initial disturbance vectors. The constant matrices $A_k \in R^{n\times n}$ characterize the physical properties of the system such as inertia, Coriolis acceleration, damping, and stiffness.[5,6] The nonnegative T_d is a time delay parameter, and E_k are $n \times n$ constant matrices.

Existing modal analysis techniques are not capable of obtaining a closed-form solution for such a delayed system. To see this, consider the eigenvalue problem associated with Eq. (1),

$$\left(A_2\lambda^2 + A_1\lambda + A_0\right)\boldsymbol{u} + e^{-\lambda T_d}(E_1\lambda + E_0)\boldsymbol{u} = 0 \quad (3)$$

where λ and $\boldsymbol{u}$ are the eigenvalue and corresponding eigenvector, respectively. Because of the time delay, an orthogonality relation for the eigensolutions of Eq. (3) is difficult to find. As such, conventional modal analyses fail to decouple Eq. (1). In practice, Eq. (1) is solved by approximate methods, such as direct time-integration procedures and finite difference methods.[11]

On the other hand, the solution to Eq. (1) in theory can be expressed by Laplace transform without the need for orthogonal eigensolutions. To this end, take the Laplace transform of Eq. (1),

$$Z(s)\hat{\boldsymbol{y}}(s) = \hat{\boldsymbol{f}}(s) + \hat{\boldsymbol{f}}_I(s) \quad (4)$$

where the caret indicates Laplace transformation and the $n \times n$ complex matrix $Z(s)$ is defined as the impedance matrix of the system and is given by

$$Z(s) = A_2s^2 + A_1s + A_0 + (E_1s + E_0)\,e^{-T_ds} \quad (5)$$

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and the vector $\hat{\boldsymbol{f}}_I(s)$ is related to the initial disturbances by $\hat{\boldsymbol{f}}_I(s) = (A_2 s + A_1)\boldsymbol{a}_0 + A_2\boldsymbol{b}_0$. By inverse Laplace transform, the time-domain response is obtained as

$$\boldsymbol{y}(t) = \mathcal{L}^{-1}[\hat{H}(s)(\hat{\boldsymbol{f}}(s) + \hat{\boldsymbol{f}}_I(s))] \tag{6}$$

where

$$\hat{H}(s) = Z^{-1}(s) \tag{7}$$

is the system transfer function. Although Eq. (6) symbolically represents an exact solution, its utility in numerical simulation of structural systems of many DOFs is limited. Because the transfer function $\hat{H}(s)$ for such a system is often obtained numerically from matrix inversion, which becomes singular at its poles (the system eigenvalues), accurate estimation of the system response by Eq. (6) is impractical, if not impossible.

The objective of the current investigation is to develop a closed-form series solution method for the delayed structural system (1) subject to arbitrary external and initial disturbances.

III. ZPM

In the proposed ZPM, a closed-form series solution of Eq. (1) is obtained from inverse Laplace transform. A major development is to establish a relation between the transfer function residues and eigensolutions of the structural system. This relation transforms the inverse Laplace transform process into determination of system eigensolutions. Many methods for accurate evaluation of the eigensolutions of structural systems are available, see Ref. 6, for instance. The closed-form series solution then can be systematically and accurately determined, without the need for orthogonal eigensolutions. The proposed method is named ZPM after a differential operation on the impedance $Z(s)$ defined in Eq. (5).

For the convenience of subsequent analysis, Eq. (6) is rewritten as

$$\boldsymbol{y}(t) = \int_0^t H(t-\tau)\boldsymbol{f}(\tau)\,\mathrm{d}\tau + H(t)(A_1\boldsymbol{a}_0 + A_2\boldsymbol{b}_0) + \frac{\mathrm{d}H(t)}{\mathrm{d}t}A_2\boldsymbol{a}_0 \tag{8}$$

where convolution integral and the initial conditions (2) have been used and Green's function $H(t)$ is the inverse Laplace transform of the system transfer function

$$H(t) = \mathcal{L}^{-1}[\hat{H}(s)] = \mathcal{L}^{-1}[Z^{-1}(s)] \tag{9}$$

Given a specific system, if its Green's function is known, its time-domain response to arbitrary external and initial disturbances is completely determined by Green's function formula, Eq. (8).

Eigenvalue Problems

In this study, Green's function in Eq. (8) is obtained in closed-form modal series. To this end, two sets of eigenvalue problems are defined. The first set is the associate and adjoint eigenvalue problems (EVPs) of the original governing equation, Eq. (1).

EVP1:

$$Z(\lambda_k)\boldsymbol{u}_k = 0, \qquad k = 1, 2, \ldots \tag{10a}$$

EVP1a:

$$Z^T(\lambda_k)\boldsymbol{v}_k = 0, \qquad k = 1, 2, \ldots \tag{10b}$$

where the eigenvalues λ_k are the roots of the characteristic equation

$$\det[Z(s)] = 0 \tag{11}$$

and Z^T is the transpose of Z. The λ_k are also the poles of the transfer function $\hat{H}(s)$. Because of the time delay, there is an infinite number of eigenvalues although the system has a finite number of DOFs. Among the infinite eigenvalues, $2n$ are related to the n modes of vibration of the structural system; the rest are induced by the time delay.[15]

Now, define the second set of EVPs. Given a value of the complex parameter s, the associate and adjoint EVPs of the impedance matrix $Z(s)$ are as follows:

EVP2:

$$Z(s)\phi_j(s) = \mu_j(s)\phi_j(s), \qquad j = 1, 2, \ldots, n \tag{12a}$$

EVP2a:

$$Z^T(s)\psi_j(s) = \mu_j(s)\psi_j(s), \qquad j = 1, 2, \ldots, n \tag{12b}$$

where the eigenvalues $\mu_j(s)$ are the roots of the characteristic equation

$$\det[Z(s) - \mu I] = 0 \tag{13}$$

The symbols $\mu_j(s)$, $\phi_j(s)$, and $\psi_j(s)$ indicate the dependence of the eigensolutions on the value of s. Because s is fixed, the number of the eigenvalues in EVP2 and EVP2a is always n, the dimension of the matrix $Z(s)$.

The λ_k of EVP1 shall be called the system eigenvalues, and $\mu_j(s)$ of EVP2 called the impedance eigenvalues. Without loss of generality assume that the system eigenvalues are distinct. (Repeated eigenvalues are discussed in Appendix A.) It is easy to see from Eqs. (10) and (12) that as s approaches an eigenvalue of EVP1, for example, λ_k, at least and at most one eigenvalue of EVP2 vanishes; that is, $\det[Z(\lambda_k) - \mu I] = 0$ has one and only one zero root. Denote this eigenvalue and the corresponding eigenvectors by $\mu_1(s)$, $\phi_1(s)$, and $\psi_1(s)$, respectively. It is easy to show the following properties:

$$\mu_1(\lambda_k) = 0, \qquad \phi_1(\lambda_k) = \boldsymbol{u}_k, \qquad \psi_1(\lambda_k) = \boldsymbol{v}_k \tag{14a}$$

and

$$\mu_j(\lambda_k) \neq 0, \qquad \text{for} \qquad j = 2, \ldots, n \tag{14b}$$

where $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ are the eigenvectors of EVP1 and EVP1a corresponding to λ_k. The $\mu_1(s)$ shall be called the root locus originating from the eigenvalue λ_k.

Transfer Function Residues

The system eigenvalues λ_k in most engineering applications form a countable set. According to the residue theorem,[16] Green's function is expressed by

$$H(t) = \sum_{k=1}^{\infty} e^{\lambda_k t} \operatorname*{res}_{s=\lambda_k}[\hat{H}(s)] \tag{15}$$

where the residue of the transfer function at the pole λ_k is given by

$$\operatorname*{res}_{s=\lambda_k}[\hat{H}(s)] = \lim_{s\to\lambda_k}(s-\lambda_k)\hat{H}(s) \tag{16}$$

Although the formula (16) is well known, it is rather theoretical and is limited to simple problems. The transfer functions of many complex systems can only be determined numerically through matrix inversion, i.e., $\hat{H}(s) = Z^{-1}(s)$. Because of the singularities of the transfer function at its poles, accurate evaluation of the transfer function residues by Eq. (16) is very difficult. In this subsection, the transfer function residues are evaluated by a new approach that avoids matrix inversion and transfer function singularities.

Claim 1. For s in a small neighborhood of λ_k in the complex plane, the transfer function of the dynamic system (1) is of the form

$$\hat{H}(s) = \frac{1}{\mu_1(s)\psi_1^T(s)\phi_1(s)}\phi_1(s)\psi_1^T(s) + R_a(s) \tag{17}$$

where $\mu_1(s)$, $\phi_1(s)$, and $\psi_1(s)$ are defined in Eq. (12), and $R_a(s)$ is an $n \times n$ complex matrix being finite and analytic at λ_k.

Proof. See Appendix B.

Claim 1 indicates that the singularity of the system transfer function around λ_k is completely determined by $\mu_1(s)$, $\phi_1(s)$, and $\psi_1(s)$.

Claim 2. The transfer function residue at λ_k is given by

$$\operatorname*{res}_{s=\lambda_k}[\hat{H}(s)] = \frac{1}{\sigma_k \boldsymbol{v}_k^T\boldsymbol{u}_k}\boldsymbol{u}_k\boldsymbol{v}_k^T \tag{18}$$

where $\sigma_k = \mathrm{d}\mu_1(s)/\mathrm{d}s|_{s=\lambda_k}$, which is the sensitivity coefficient of the root locus $\mu_1(s)$ at λ_k.

Proof. When s is near λ_k, the root locus by Taylor expansion is

$$\mu_1(\lambda_k + \varepsilon) = \mu_1(\lambda_k) + \frac{d}{ds}\mu_1(s)\bigg|_{s=\lambda_k} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = \sigma_k\varepsilon + \mathcal{O}(\varepsilon^2)$$

where $\varepsilon = s - \lambda_k$ and Eqs. (14a) have been used. By Claim 1, the transfer function becomes

$$\hat{H}(s) = \frac{1}{\left[\sigma_k\varepsilon + \mathcal{O}(\varepsilon^2)\right]\psi_1^T(s)\phi_1(s)}\phi_1(s)\psi_1^T(s) + R_a(s)$$

Substituting the preceding equation into Eq. (16) and letting ε go to zero yields Eq. (18). □

Theorem 1. The transfer function residue at λ_k is given by

$$\operatorname*{res}_{s=\lambda_k}[\hat{H}(s)] = \frac{1}{v_k^T Z'(\lambda_k)u_k}u_k v_k^T \tag{19}$$

where $Z'(s) = dZ(s)/ds$.

Proof. Differentiating Eq. (12a) with respect to s and premultiplying the resulting equation by $\psi_1^T(s)$ gives

$$\psi_1^T(s)Z'(s)\phi_1(s) + [Z(s)\psi_1(s)]^T\phi_1'(s)$$
$$= \mu_1'(s)\psi_1^T(s)\phi_1(s) + \mu_1(s)\psi_1^T(s)\phi_1'(s)$$

where the prime stands for d/ds. By Eqs. (10a) and (14), as $s \to \lambda_k$, the preceding equation reduces to

$$\sigma_k = \frac{d}{ds}\mu_1(s)\bigg|_{s=\lambda_k} = \frac{\boldsymbol{v}_k^T Z'(\lambda_k)\boldsymbol{u}_k}{\boldsymbol{v}_k^T\boldsymbol{u}_k} \tag{20}$$

which, by Claim 2, yields Eq. (19). □

The derivative $Z'(s)$ in Eq. (19) can be easily computed without involving any transfer function singularities or matrix inversion:

$$Z'(s) = 2sA_2 + A_1 + (E_1 - T_d E_0 - sT_d E_1)e^{-T_d s} \tag{21}$$

Theorem 1 relates the transfer function residues to the system eigensolutions, which, as shall be seen, provides a new approach to closed-form transient analysis. Although the theorem does not advance any solution technique for the eigenvalue problems EVP1 and EVP1a, many efficient numerical schemes are available for estimation of system eigensolutions. Hence, with the residue-eigensolution relation, Eq. (19), accurate evaluation of the transfer function residues is achievable.

Note that the derivation of the root locus sensitivity coefficient given in Eq. (20) is totally different from a sensitivity analysis in structural dynamics. The latter considers the variation of the system eigenvalue $\lambda_k(p)$ due to the change of a physical parameter p, e.g., inertia, damping, or stiffness; the former examines the trajectory of the impedance eigenvalue $\mu_1(s)$ vs the complex parameter s. Unlike $\lambda_k(p)$ and p, here $\mu_1(s)$ and s have no specific physical meaning.

Closed-Form Modal Expansion

Combining Theorem 1 and Green's function formula (8), one immediately arrives at the following modal expansion theorem:

Theorem 2. Green's function and transient response of the delayed structural system (1) are expressed by the closed-form modal series

$$H(t) = \sum_{k=1}^{\infty} e^{\lambda_k t}\frac{1}{\boldsymbol{v}_k^T Z'(\lambda_k)\boldsymbol{u}_k}\boldsymbol{u}_k\boldsymbol{v}_k^T \tag{22}$$

and

$$\boldsymbol{y}(t) = \sum_{k=1}^{\infty}\frac{q_k(t)}{\boldsymbol{v}_k^T Z'(\lambda_k)\boldsymbol{u}_k}\boldsymbol{u}_k \tag{23}$$

where λ_k, $\boldsymbol{u}_k$, and $\boldsymbol{v}_k$ are the eigensolutions of EVP1 and EVP1a and $Z'(\lambda_k)$ by Eq. (21) is

$$Z'(\lambda_k) = 2\lambda_k A_2 + A_1 + (E_1 - T_d E_0 - \lambda_k T_d E_1)e^{-T_d\lambda_k} \tag{24}$$

and the time-dependent coefficients $q_k(t)$ are

$$q_k(t) = \int_0^t e^{\lambda_k(t-\tau)}\boldsymbol{v}_k^T\boldsymbol{f}(\tau)\,d\tau + e^{\lambda_k t}\boldsymbol{v}_k^T(A_1\boldsymbol{a}_0 + A_2\boldsymbol{b}_0 + \lambda_k A_2\boldsymbol{a}_0) \tag{25}$$

In the preceding modal expansion, no orthogonality relations among system eigenvectors have been used. In other words, the ZPM-based modal expansion does not need to decouple the original equation of motion. In addition, $q_k(t)$ are given by exact and explicit quadrature.

This result is not available either in standard inverse Laplace transform technique or in a conventional modal analysis. It is the residue-eigensolution relation (19) that links the two methods such that the closed-form modal series solution is obtained.

As a special example, when the time delay is zero, the system has $2n$ eigenvalues and Eq. (23) reduces to

$$\boldsymbol{y}(t) = \sum_{k=1}^{2n}\frac{q_k(t)}{\boldsymbol{v}_k^T\{2\lambda_k A_2 + A_1 + E_1\}\boldsymbol{u}_k}\boldsymbol{u}_k$$

which is equivalent to the result obtained by using biorthogonal eigenvectors.[17] The difference is that the ZPM does not need any orthogonality relations.

IV. Example

The proposed ZPM is illustrated on a gyroscopic system under delayed velocity feedback control. In Fig. 1, a mass m is connected by four springs of identical stiffness $k/2$ to the frame oy_1y_2 that rotates with respect to the fixed frame OXY, with a constant speed Ω. A control force $f_c(t)$, proportional to the velocity of the mass in the y_1 direction, is applied to the mass in the y_2 direction. The governing equation of the system in the rotational coordinates is

$$\left(\begin{bmatrix} m & 0 \\ 0 & m\end{bmatrix}\frac{d^2}{dt^2} + \begin{bmatrix} 0 & -2\Omega m \\ 2\Omega m & 0\end{bmatrix}\frac{d}{dt} + \begin{bmatrix} k - m\Omega^2 & 0 \\ 0 & k - m\Omega^2\end{bmatrix}\right) \times \begin{pmatrix} y_1(t) \\ y_2(t)\end{pmatrix} = \begin{pmatrix} p_1(t) \\ f_c(t) + p_2(t)\end{pmatrix} \tag{26}$$

where $p_1(t)$ and $p_2(t)$ are the external forces (not shown in Fig. 1) and the control force is

$$f_c(t) = -g_c\frac{d}{dt}y_1(t - T_d) \tag{27}$$

with g_c being a control gain and T_d a nonnegative time delay constant. According to Eq. (27), the controller transfer function is

$$G_c(s) = \hat{f}_c(s)/\hat{y}_1(s) = -g_c s e^{-T_d s}$$

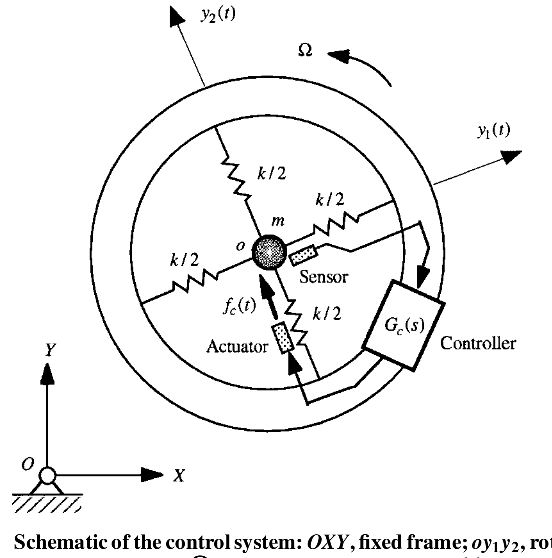

Fig. 1 Schematic of the control system: OXY, fixed frame; oy_1y_2, rotating frame, of rotation speed Ω with respect to OXY; and $G_c(s)$, controller transfer function.

The impedance matrix of the controlled gyroscopic system is

$$Z(s) = \begin{bmatrix} ms^2 + k - m\Omega^2 & -2\Omega ms \\ 2\Omega ms + g_c s e^{-T_d s} & ms^2 + k - m\Omega^2 \end{bmatrix} \tag{28}$$

The characteristic equation of the control system is $\det Z(s) = 0$, or in an explicit form

$$(s^2 + \alpha^2 - \Omega^2)^2 + 2\Omega^2 s^2 (2 + \mu e^{-T_d s}) = 0 \tag{29}$$

where $\alpha = \sqrt{(k/m)}$ and $\mu = g_c/(m\Omega)$. The explicit form (29) is given for the purpose of presentation; it is not needed in numerical simulation. The derivative of $Z(s)$ is

$$Z'(s) = \begin{bmatrix} 2ms & -2\Omega m \\ 2\Omega m + g_c(1 - T_d s)e^{-T_d s} & 2ms \end{bmatrix} \tag{30}$$

The parameters of the gyroscopic system are chosen as $m = 1$, $k = 100$, $\Omega = 2$, indicating that $\alpha = \sqrt{(k/m)} = 10$. Three cases are considered in the numerical simulation: case 1, no control, $\mu = 0$; case 2, nondelayed velocity feedback control, $T_d = 0$ and $\mu = 1.0$; and case 3, delayed velocity feedback control, $T_d = 5\pi/48 = 0.3272$ and $\mu = 1.0$. Here $\mu = 1.0$ in cases 2 and 3 indicate that $g_c = m\Omega\mu = 2.0$.

In case 1, the gyroscopic system has four imaginary eigenvalues: $\pm j8$ and $\pm j12$. In case 2, the system eigenvalues are still imaginary: $\pm j7.65$ and $\pm j12.55$. In fact, for any gain $\mu > 0$, the system eigenvalues in case 2 are always imaginary. This implies that the nondelayed velocity feedback does not stabilize the gyroscopic system, due to dislocation of the sensor and actuator.

In case 3, the first 12 pairs of eigenvalues of the controlled gyroscopic system are computed by using MATLAB software and are listed in Table 1. The first two pairs of eigenvalues are related the two modes of vibration of the gyroscopic system, whereas the remaining 10 are among the infinite number of eigenvalues induced by the time delay. It can be shown that, for $0 < \mu < 1.98$, all of the eigenvalues λ_k of the controlled gyroscopic system have negative real part and that $\mathrm{Re}(\lambda_{k+1}) < \mathrm{Re}(\lambda_k) < 0$ for $k = 1, 2, \ldots$. The first four eigenvalue loci $s_k(\mu)$ of the control system vs the gain parameter μ are plotted in Fig. 2, where s_1 and s_2 are related to the two modes of vibration, whereas s_3 and s_4 are induced by the time delay.

The free response of the gyroscopic system under the initial conditions $y_1(0) = 0$, $y_2(0) = -0.4$, $\dot{y}_1(0) = 0$, and $\dot{y}_2(0) = 0$ are shown in Figs. 3–5 for the cases 1–3, respectively. In case 3, the first eight modes, i.e., the first eight pairs of eigenvalues, are used in computation. Figure 5 shows that the time delay helps stabilize the system response.

To verify the convergence of the ZPM, the displacement $y_2(t)$ in case 3 is plotted in Fig. 6 for three models: $n = 2$, 3, and 4, where n is the number of pairs of eigenvalues used in computation. For $t > 0.3$, all three models are in good agreement. However, when t is small, deviations among the models become obvious, and the largest deviation occurs at $t = 0$. For $n = 2$, which only models the two vibration modes of the gyroscopic system, the computed response does not satisfy the initial condition $y_2(0) = -0.4$. This means that the contributions of those time-delay-induced modes can be significant at earlier times and must be included for accurate estimation.

Table 1 Eigenvalues of the gyroscopic system under the delayed velocity feedback ($J = \sqrt{-1}$)

Mode number k	λ_k
1	$-0.2336 \pm J8.4625$
2	$-0.5036 \pm J11.394$
3	$-11.043 \pm J12.563$
4	$-15.732 \pm J35.523$
5	$-18.363 \pm J55.534$
6	$-20.165 \pm J75.144$
7	$-21.545 \pm J94.602$
8	$-22.665 \pm J113.98$
9	$-23.610 \pm J133.32$
10	$-24.426 \pm J152.62$
11	$-25.146 \pm J171.91$
12	$-25.789 \pm J191.18$

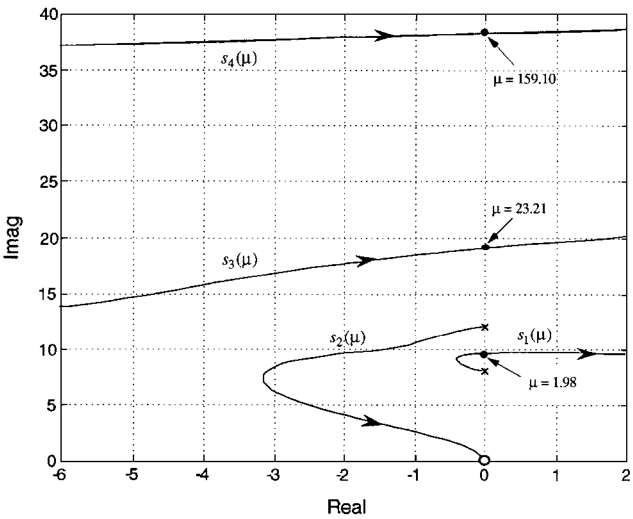

Fig. 2 Eigenvalue loci $s_k(\mu)$ of the gyroscopic system under delayed velocity feedback: ○, open-loop zero, and ×, open-loop pole.

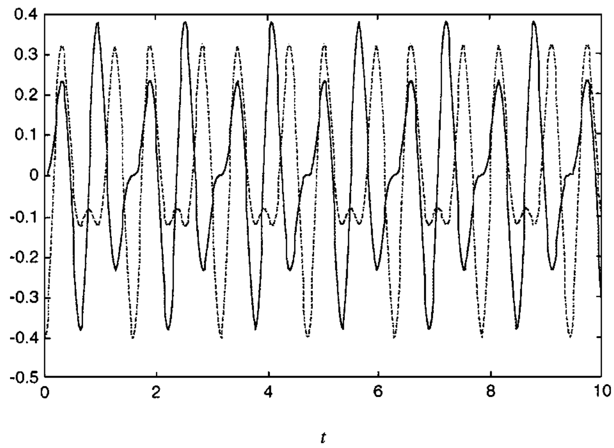

Fig. 3 Free response of the gyroscopic system without control (case 1): ——, $y_1(t)$, and - - - -, $y_2(t)$.

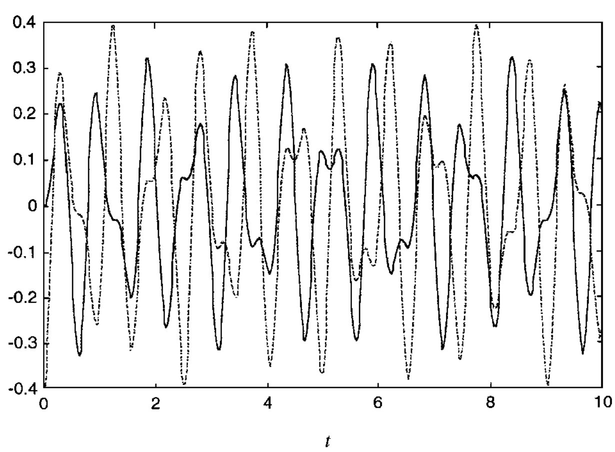

Fig. 4 Free response of the gyroscopic system under nondelayed velocity feedback (case 2): ——, $y_1(t)$, and - - - -, $y_2(t)$.

As n increases, the error dramatically decreases, as justified by the curves of $n = 3$ and 4. For $n = 4$, the simulated response has an error of 0.5% at $t = 0$. The model used to generate Fig. 5 has eight modes ($n = 8$), with an error of 0.035% at $t = 0$.

The efficiency of the proposed method is demonstrated in Fig. 7, where the displacement $y_1(t)$ of case 3 is obtained by the ZPM and a finite difference algorithm given in Appendix C. The ZPM model consists of eight modes ($n = 8$), which has been shown to

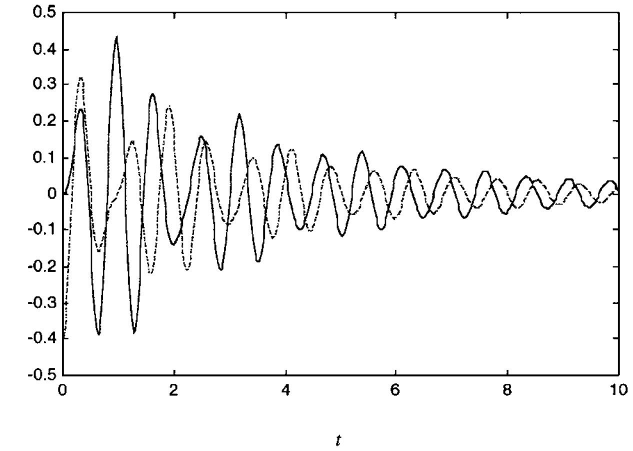

Fig. 5 Free response of the gyroscopic system under delayed velocity feedback (case 3): ——, $y_1(t)$, and - - - -, $y_2(t)$.

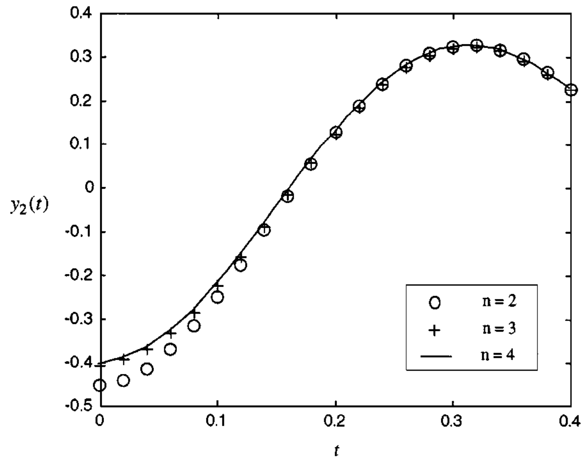

Fig. 6 Convergence of the ZPM-based prediction (case 3): n, number of modes (pairs of eigenvalues) used in simulation.

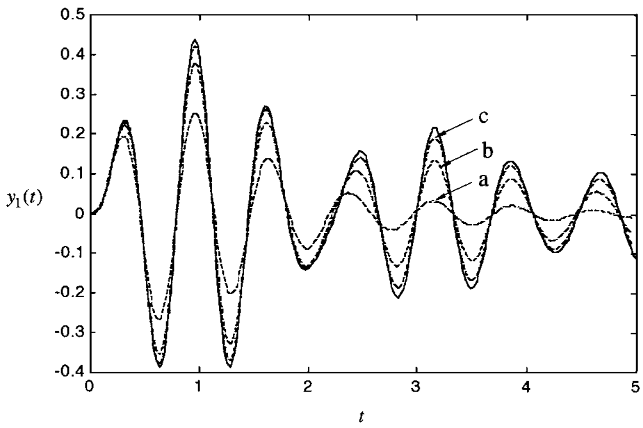

Fig. 7 Free response of the gyroscopic system under delayed velocity feedback (case 3): ——, ZPM ($n = 8$), and - - - -, finite difference method; a, $\Delta t = \pi/240$; b, $\Delta t = \pi/960$; and c, $\Delta t = \pi/3840$.

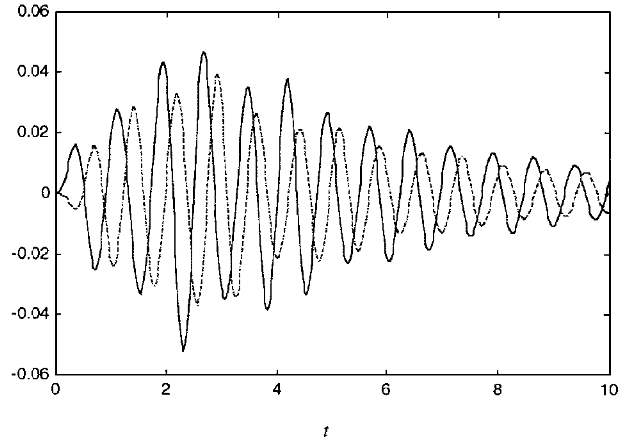

Fig. 8 Forced response of the gyroscopic system under delayed velocity feedback (case 3): ——, $y_1(t)$, and - - - -, $y_2(t)$.

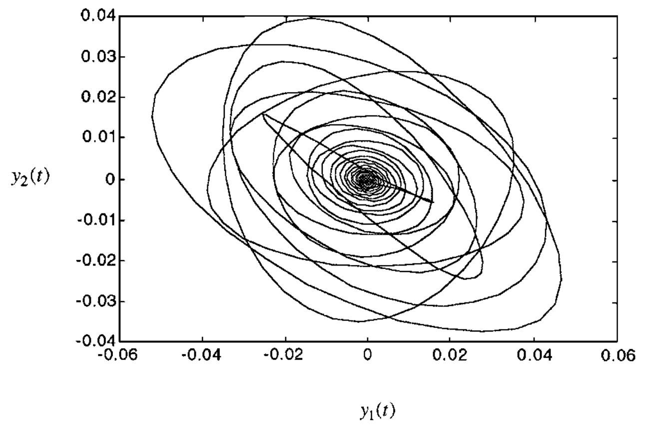

Fig. 9 Trajectory of the gyroscopic system under delayed velocity feedback in the rotating frame oy_1y_2, $0 \leq t \leq 50$.

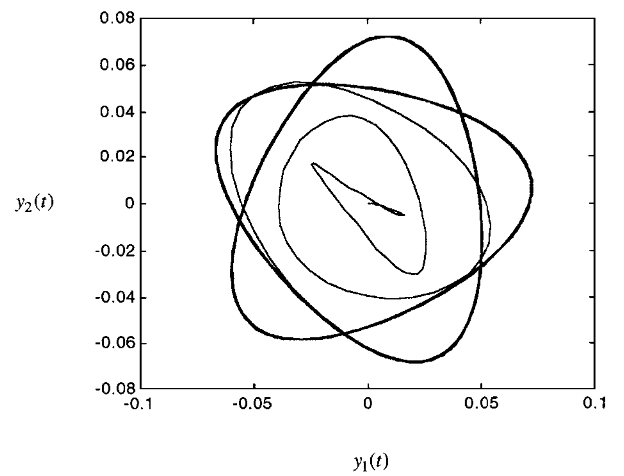

Fig. 10 Trajectory of the uncontrolled gyroscopic system in the rotating frame oy_1y_2, $0 \leq t \leq 50$.

be accurate. The finite difference solution converges to the ZPM solution as the time step Δt shrinks. However, the errors in the finite difference iteration accumulate as time passes, yielding an increased deviation between the finite difference solution and the ZPM solution. The finite difference method demands substantial computational efforts: a very small time step $\Delta t = \pi/3840 = 8.18 \times 10^{-4}$ (more than 1200 points within unit time) gives only relatively fine results (curve c in Fig. 7); a time step $\Delta t = \pi/960 = 0.00327$, which is still small, yields unacceptable results (curve b). And this is just for a 2-DOF system. On the other hand, the ZPM-based analysis needs only to evaluate an eight-term truncated modal series from Eq. (23), which is simple, and does not requires simulation at a large number of points in time.

Finally, determine the forced response of the gyroscopic system under the external disturbance

$$p_1(t) = \begin{cases} \sin 7.999t & \text{for} \quad 0 \leq t \leq 2.5 \\ 0 & \text{for} \quad t > 2.5 \end{cases}$$

$$p_2(t) = 0$$

and zero initial disturbances. The excitation frequency is very close to the first natural frequency of the uncontrolled gyroscopic system.

The transient response of the gyroscopic system under delayed feedback control (case 3) is obtained by the ZPM with $n = 8$; see Fig. 8. The trajectory of the lumped mass in the rotating frame $o\boldsymbol{y}_1\boldsymbol{y}_2$ is shown in Fig. 9 for $0 \le t \le 50$. Because of the active damping provided by the delayed feedback, the mass eventually settles at the origin. For comparison, the trajectory of the uncontrolled lumped mass (case 1) is given in Fig. 10, showing that the steady-state motion of the mass is periodic and nonzero.

V. Conclusions

The ZPM presented is capable of obtaining closed-form modal series solutions for delayed structural systems, for which conventional modal analyses fail. One important discovery is that the transfer function residues of the delayed structural system are representable by their eigensolutions. Because of this relation, orthogonality among the system eigenvectors is not required in a ZPM-based transient analysis. The numerical example justifies the accuracy and efficiency of the proposed method. Although only lumped parameter models of structural dynamic systems are examined, the proposed ZPM can be extended to distributed parameter models governed by partial differential equations.

Appendix A: Systems with Repeated Eigenvalues

If the structural system described by Eq. (1) has repeated eigenvalues, the associate and adjoint eigenvalue problems are defined as follows:

EVP1:

$$Z(\lambda_k)\boldsymbol{u}_{kj} = 0, \qquad j = 1, 2, \ldots, m_k \tag{A1}$$

EVP1a:

$$Z^T(\lambda_k)\boldsymbol{v}_{kj} = 0, \qquad j = 1, 2, \ldots, m_k \tag{A2}$$

where m_k is the multiplicity of the eigenvalues λ_k. These repeated eigenvalues in most engineering applications are related to the modes of vibration in the structural system, but not those induced by the time delay. Hence, $\boldsymbol{u}_{kj}$ and $\boldsymbol{v}_{kj}$ are linearly independent eigenvectors describing different mode shapes. Following the proof of Theorem 1, it can be shown that the transfer function residue at λ_k is given by

$$\operatorname*{res}_{s=\lambda_k}[\hat{H}(s)] = \sum_{j=1}^{m_k} \frac{1}{\sigma_{kj}\boldsymbol{v}_{kj}^T\boldsymbol{u}_{kj}}\boldsymbol{u}_{kj}\boldsymbol{v}_{kj}^T \tag{A3}$$

where

$$\sigma_{kj} = \frac{\boldsymbol{v}_{kj}^T Z'(\lambda_k)\boldsymbol{u}_{kj}}{\boldsymbol{v}_{kj}^T\boldsymbol{u}_{kj}}, \qquad j = 1, 2, \ldots, m_k \tag{A4}$$

Thus, by Eq. (15), Green's function of the structural system is

$$H(t) = \sum_k \sum_{j=1}^{m_k} e^{\lambda_k t}\frac{1}{\boldsymbol{v}_{kj}^T Z'(\lambda_k)\boldsymbol{u}_{kj}}\boldsymbol{u}_{kj}\boldsymbol{v}_{kj}^T \tag{A5}$$

The system transient response can be similarly obtained by following Eqs. (23) and (25).

Appendix B: Proof of Claim 1

By Eq. (14), for s in a small neighborhood of λ_k, $\mu_1(s) \neq \mu_j(s)$ for $j = 2, \ldots, n$, which implies that $\psi_1^T(s)\phi_j(s) = 0$ and $\phi_1^T(s)\psi_j(s) = 0$ for $j = 2, \ldots, n$. Note that among $\mu_2(s)$, $\mu_3(s), \ldots, \mu_n(s)$, there may be repeated eigenvalues. The following two equations hold true:

$$Z(s)\Phi(s) = \Phi(s)\begin{bmatrix} \mu_1(s) & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & J_R(s) \end{bmatrix} \tag{B1}$$

$$\Psi^T(s)\Phi(s) = \begin{bmatrix} \psi_1^T(s)\phi_1(s) & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & \Delta_R(s) \end{bmatrix} \tag{B2}$$

where $\Phi = [\phi_1 \vdots \Phi_R] \in C^{n\times n}$ and $\Psi = [\psi_1 \vdots \Psi_R] \in C^{n\times n}$, with Φ_R, $\Psi_R \in C^{n\times(n-1)}$ consisting of the eigenvectors ϕ_j and ψ_j, $j = 2, 3, \ldots, n$; $0_{1\times(n-1)}$ and $0_{(n-1)\times 1}$ are zero matrices with the dimensions indicated by their subscripts; $J_R(s)$ is a Jordan matrix associated with the eigenvalues $\mu_j(s)$, $j = 2, 3, \ldots, n$; and $\Delta_R(s) = \Psi_R^T(s)\Phi_R(s)$. It follows that

$$Z^{-1}(s) = [\phi_1 \vdots \Phi_R]\begin{bmatrix} \dfrac{1}{\mu_1(s)} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & J_R^{-1}(s) \end{bmatrix}\Phi^{-1}(s) \tag{B3}$$

and

$$\Phi^{-1}(s) = \begin{bmatrix} \dfrac{1}{\Psi_1^T(s)\phi_1(s)} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & \Delta_R^{-1}(s) \end{bmatrix}[\psi_1 \vdots \Psi_R]^T \tag{B4}$$

Substituting Eq. (B4) into Eq. (B3) yields Eq. (17), with the matrix $R_a(s)$ given by

$$R_a(s) = \Phi_R(s)J_R^{-1}(s)\Delta_R^{-1}(s)\Psi_R^T(s) \tag{B5}$$

Because of Eq. (14b), $J_R^{-1}(s)$ is finite and analytic at λ_k and so is $R_a(s)$.

Appendix C: Finite Difference Solution for the Example in Section IV

The controlled gyroscopic system by Eqs. (26) and (27) is governed by

$$M\ddot{\boldsymbol{y}}(t) + G\dot{\boldsymbol{y}}(t) + K\boldsymbol{y}(t) + E_1\dot{\boldsymbol{y}}(t - T_d) = \boldsymbol{p}(t) \tag{C1}$$

where

$$\boldsymbol{y}(t) = [\boldsymbol{y}_1(t) \quad \boldsymbol{y}_2(t)]^T, \qquad \boldsymbol{p}(t) = [p_1(t) \quad p_2(t)]^T$$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & -2\Omega m \\ 2\Omega m & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k - m\Omega^2 & 0 \\ 0 & k - m\Omega^2 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0 & 0 \\ g_c & 0 \end{bmatrix}$$

The derivatives in Eq. (C1) are approximated by

$$\dot{\boldsymbol{y}}(t_k) \approx \frac{1}{\Delta t}(\boldsymbol{y}_k - \boldsymbol{y}_{k-1}), \qquad \ddot{\boldsymbol{y}}(t_k) \approx \frac{1}{(\Delta t)^2}(\boldsymbol{y}_k - 2\boldsymbol{y}_{k-1} + \boldsymbol{y}_{k-2}) \tag{C2}$$

where $\boldsymbol{y}_k = \boldsymbol{y}(t_k)$ and $t_k = k\Delta t$, $k = 1, 2, \ldots$; the time step Δt is chosen such that $\Delta t = T_d/n_d$ or $T_d = n_d\Delta t$, with n_d being a positive integer. Substitute Eq. (C2) into Eq. (C1) to obtain the finite difference solution algorithm

$$\boldsymbol{y}_k = B_1\boldsymbol{y}_{k-1} + B_2\boldsymbol{y}_{k-2} + B_d(\boldsymbol{y}_{k-n_d} - \boldsymbol{y}_{k-n_d-1}) + q_k \tag{C3}$$

for $k = 1, 2, \ldots,$ where $B_1 = H(2M + \Delta tG)$, $B_2 = -HM$, $B_d = -\Delta t HE_1$, $q_k = (\Delta t)^2 H\boldsymbol{p}(t_k)$, and $H = [M + \Delta tG + (\Delta t)^2 K]^{-1}$. By Eqs. (2), the initial conditions for the algorithm are

$$\boldsymbol{y}_0 = \boldsymbol{a}_0, \qquad \boldsymbol{y}_{-1} = \boldsymbol{a}_0 - \Delta t\boldsymbol{b}_0 \tag{C4}$$

The time delay related term in Eq. (C3)

$$B_d(\boldsymbol{y}_{k-n_d} - \boldsymbol{y}_{k-n_d-1}) = 0 \qquad \text{for} \qquad k = 1, 2, \ldots, n_d - 1 \tag{C5}$$

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